Engineering Notes

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Discrete Optimal Tracking with Prescribed Disturbances

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I. Introduction

INHOMOGENEOUS control problems appear often in robotics, in the presence of prescribed load variations. The same sort of difficulty may be faced while dealing with the flight control of a preprogrammed thrusting vehicle. In Ref. 1, the inhomogeneous terminal control problem has been solved for the continuous time deterministic case, using calculus of variations. In the present Note we broaden the discussion by the inclusion of required tracking tasks, as well as the presence of process noise terms. Formulating the problem in the discrete domain enables straightforward solution via dynamic programming[2] that yields a feedback-feedforward digital control.

II. Problem Formulation

Consider the linear discrete-time, time-varying process, in the state vector X:

$$X(k+1) = F_k X(k) + G_k U(k) + f_d(k) + W(k) \tag{1}$$

where we use the forms $(\cdot)_k$ and $(\cdot)(k)$ alternatively for the value of $(\cdot)$ at the kth stage, and all the variables are the usual vectors and matrices with appropriate dimensions. Also, $f_d(k) = f_{d,k}$ = prescribed disturbance vector function, and $w(k)$ = zero-mean random sequence with $E(w_k w_k^T k) = Q_k$. Subject to Eq. (1), we wish to minimize the cost J

$$J = E\Big\{\frac{1}{2}(X_N - f_{r,N})^T S_f (X_N - f_{r,N}) + \sum_{k=1}^{N-1}\Big[\frac{1}{2}(X_k - f_{r,k})^T A_k (X_k - f_{r,k}) + \frac{1}{2}U_k{}^T B_k U_k\Big] | X_k\Big\} \tag{2}$$

by an appropriate selection of the control vector U at each stage. In Eq. (2), S_f is a prescribed symmetrical matrix, A_k and B_k are symmetrical matrices, prescribed for all the stages, and $f_{r,k}$ is the tracking reference vector function, prescribed at any stage. We must regard the cost as an expectation, which is, however, conditioned on the state, since we assume throughout this Note that we have a complete nonanticipative knowledge of the state. Denoting the cost to go from the Kth state by J_k, the minimal cost to go J_k^* gives rise to the dynamic programming equation (DPE)[2]:

$$J_k^* = \underset{U(k)}{MIN}\Big\{E\Big[\frac{1}{2}(X_k - f_{r,k})^T A_k (X_k - f_{r,k}) + \frac{1}{2}U_k^T B_k U_k + J_{k+1}^*\Big] | X_k\Big\} \tag{3}$$

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The control $U^*(k)$ that minimizes the right-hand side of Eq. (3) is the desired optimal control, and is included in the solution of Eq. (3) for the discrete functional J_k^*.

III. Solution

We suggest that the solution to the DPE in Eq. (3) is given by

$$J_k^* = ½ X_k^T S_k X_k + \Psi_k X_k + \Phi_k \tag{4}$$

where S_k are symmetrical matrices for all k, Ψ_k are row vectors, and Φ_k are scalars. We now substitute the candidate solution, Eq. (4), expressed at the $(k+1)$ stage via Eq. (1), back into the right-hand side of Eq. (3). Averaging with respect to w and minimizing with respect to U_k yield

$$U_k^* = -(B_k + G_k^T S_{k+1} G_k)^{-1}[G_k^T S_{k+1} \times (F_k X_k + f_{d,k}) + \Psi_{k+1} G_k] \tag{5}$$

denoted for short as

$$U_k^* = -C_k X_k - d_k f_{d,k} - g_k \tag{6}$$

where:

$$C_k = (B_k + G_k^T S_{k+1} G_k)^{-1} G_k^T S_{k+1} F_k$$
$$d_k = (B_k + G_k^T S_{k+1} G_k)^{-1} G_k^T S_{k+1}$$
$$g_k = (B_k + G_k^T S_{k+1} G_k)^{-1} \Psi_{k+1} G_k \tag{7}$$

We now substitute the optimal control equation (6) into Eq. (3). (For shorter notation the subscript k will be omitted.)

$$\begin{aligned} ½X^T SX + \Psi X + \Phi = E\{ & ½(X - f_r)^T A (X - f_r) \\ & + ½(-CX - df_d - g)^T B(-CX - df_d - g) \\ & + ½[(F - GC)X + (I - Gd)f_d - Gg + W]^T S_{k+1} \\ & \times [(F - GC)X + (I - Gd)f_d - Gg + W] + \Psi_{k+1} \\ & \times [(F - GC)X + (I - Gd)f_d - Gg + W] + \Phi_{k+1} | X\} \end{aligned} \tag{8}$$

which is an identity in X. Sufficient conditions for the validity of Eq. (8) are nontrivial equalities of the coefficients of all powers of X from both sides. Equating the coefficients of the general term $x_i x_j$ in Eq. (8), we obtain

$$S_{ij} = A_{ij} + (C^T BC)_{ij} + [(F - CG)^T S_{k+1} (F - GC)]_{ij} \tag{9}$$

where C depends on S_{k+1} by Eq. (7). Equation (9) is the classical discrete Riccati equation for linear terminal controllers. Its end conditions are given by equating Eqs. (2) and (4) for the cost of the last stage N. This yields

$$S(N) = S_f \tag{10}$$

For $A_k \neq 0$ or $B_k \neq 0$, Eq. (9) yields a nontrivial precomputable solution for S_k, for any S_f. While propagating Eq. (9)

backward, the histories of both C_k and d_k are obtained by virtue of Eq. (7). Since A and B and S_f are symmetrical, S_k is clearly symmetrical for any k. Equating the coefficients of x_i in Eq. (8), we obtain

$$\Psi_i = (-f_r^T A)_i + [(df_d + g)^T BC]_i + \{(F - GC)^T S_{k+1}[(I - Gd)f_d - Gg]\}_i + [\Psi_{k+1}(F - GC)]_i \quad (11)$$

The end conditions of Eq. (11) are obtained by equating Eqs. (2) and (4) for the N stage. This yields

$$\Psi(N) = -f_{r,N}^T S_f \quad (12)$$

With the histories of S_k, C_k, d_k already precomputed, and with g_k depending on Ψ_{k+1} and S_{k+1} by use of Eq. (7), Eq. (11) can now be propagated backward from Eq. (12) to yield nontrivial solutions for both Ψ_k and g_k. This completes the precomputations for the optimal control, Eq. (6). With a full, nonanticipative accessibility of the state, this control law consisting of two parts, can now be implemented. The first part is linear in the current state, with the gains C_k stored in memory, and thus has the form of feedback. The second part consists of precomputed functions of f_d and f_r that are prescribed; thus it can be viewed as a feed-forward term. Equation (8) forces a third regression, in the parameter Φ, that has a nontrivial solution for $Q \neq 0$. This equation is omitted from the present discussion since the optimal control is independent of w, as expected in the linear process with zero-mean noise. The existence of nontrivial solutions for the coefficients of Eq. (8) validates the proposed solution, Eq. (4), for the minimum cost functional.

IV. Alternative Formulation

It may be convenient at times to use the relative state vector X_r in the tracking problem formulation, where X_r is defined by

$$X_r(k) \triangleq X(k) - f_r(k) \quad (13)$$

Equation (1) is then equivalently described by difference equations in the relative state vector, that is

$$X_r(k+1) = F_k X_r(k) + G_k U(k) + f_d(k) + F_k f_r(k) - f_r(k+1) + W(k) \quad (14)$$

To minimize the cost in the relative coordinates J_r, we write

$$J_r = E\left\{\frac{1}{2}X_r^T(N)S_f X_r(N) + \sum_{k=1}^{N-1}\left[\frac{1}{2}X_r^T(k)A_k X_r(k) + \frac{1}{2}U^T(k)B_k U(k)\right]\Big| X_r(k)\right\} \quad (15)$$

The analysis in Secs. II and III is clearly valid for this formulation if we replace $f_d(k)$ everwhere by the expression

$$f_d(k) + F_k f_r(k) - f_r(k+1) \triangleq f_{d,r} \quad (16)$$

and cancel the term $(-f_r A)_i$ in Eq. (11).

The end conditions for the regressions of both Ψ in Eq. (11) and Φ will be zero for this case but their solutions will clearly remain nontrivial. The consequent optimal control will surely remain in the form of Eq. (6), but with $f_{d,r}$ replacing f_d and X_r replacing X, which often leads to a simpler implementation.

V. Concluding Remarks

Tracking requirements and prescribed inputs both lead to the same structure of the cost functional when they appear in linear-quadratic optimization problems. This remains true in the presence of a zero-mean process noise, enabling one to address the combination of these three classical elements in a unified and inclusive approach. Applying dynamic programming to the discrete version of the problem, we readily obtain a closed-form solution for the optimal control, which consists of two parts. The first part is the classical optimal feedback control. The second part contains two feedforward terms that are precomputed, based on the prescribed disturbance and reference.

References

[1]Garber, V., "Optimum Intercept Laws for Accelerating Targets," *AIAA Journal*, Vol. 6, No. 11, Nov. 1968, pp. 2196–2198.

[2]Bellman, R., *Dynamic Programming*, Princeton University Press, Princeton, NJ, 1957.

Molniya Orbits Obtained by the Two-Burn Method

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Introduction

SOME U.S. satellite missions require the utilization of a 12- or 24-h (period) Molniya orbit. Its inclination angle of 63.4 deg precludes rotation of the line of apsides, thus assuring that the majority of orbit time is spent in the northern or southern hemisphere, depending on where the perigee is "anchored." Launch restrictions from both ETR and WTR permit inclinations that are approximately 6½ deg above or below 63.4 deg, thus requiring a "dog-leg" maneuver, which is costly in ΔV.

Starting from a low-altitude circular orbit, a Molniya orbit can be achieved by a sequential three-burn method (adjusting apogee, perigee, and orbit inclination) or by a two-burn method, in which the first perigee burn raises apogee, and the second burn, at a true anomaly greater than 90 deg, simultaneously raises apogee and perigee to their final values, while making the required inclination change.

The purposes of this Note are threefold: 1) to demonstrate that the two-burn method is significantly more efficient than a three-burn method and is close to being the optimal solution for the cases studied; 2) to present some results that illustrate this and also permit determination of ΔVs for preliminary mission design purposes; and 3) to indicate the logic used in developing the program for IBM PC execution.

Discussion

We will assume an ELV or STS launch into a circular parking orbit. A typical ETR shuttle parking orbit is circular, at 150 n.mi. altitude, with an inclination of, at most, 57 deg. A Molniya orbit has an inclination of 63.4 deg and is highly elliptical (a typical perigee is 400 n.mi., which means, for a 12-h period, an apogee of 21,449 n.mi.) and has a perigee at the most southerly latitude.

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